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by

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UNIFORM COVERS AT NON-ISOLATED POINTS

FUCAI LIN AND SHOU LIN

Abstract. In this paper, the authors define a space with a uniform base at non-isolated points, give some characterizations of images of metric spaces by boundary-compact maps, and study certain relationships among spaces with special base properties. The main results are the following: (1) X is an open, boundary-compact image of a metric space if and only if X has a uniform base at non-isolated points; (2) each discretizable space of a space with a uniform base is an open compact and at most boundary-one image of a space with a uniform base; (3) X has a point-countable base if and only if X is a bi-quotient, at most boundary-one and countable-to-one image of a metric space.

1. Introduction

Topologists obtained many interesting characterizations of the images of metric spaces by some kind of maps. A. V. Arhangel'skiĭ [3] proved that a space X is an open compact image of a metric space if and only if X has a uniform base. Recently, Chuan Liu [16] gave a new characterization of spaces with a point-countable base by pseudo-open and at most boundary-one images of metric spaces. How could an open or pseudo-open and boundary-compact image of a metric space be characterized? On the other hand, a study of

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spaces with a sharp base or a weakly uniform base [5], [6] shows that some properties of a non-isolated point set of a topological space will help us discuss a whole construction of a space. In this paper, the authors analyze some base properties on non-isolated points of a space, introduce a space having a uniform base at non-isolated points and describe it as an image of a metric space by open boundary-compact maps. Some relationships among the images of metric spaces under open boundary-compact maps, pseudo-open boundary-compact maps, open compact maps, and spaces with a point-countable base are discussed.

By $\mathbb{R}, \mathbb{N}$, denote the set of real numbers and positive integers, respectively. For a space X, let

$$I(X) = \{x : x \text{ is an isolated point of } X\}$$

and

$$\mathcal{I}(X) = \{\{x\} : x \in I(X)\}.$$

In this paper, all spaces are T_2 and all maps are continuous and onto. Recall some basic definitions.

Let X be a topological space. X is called a *metacompact* (*para-compact*, *metalindelöf* , resp.) space if every open cover of X has a point-finite (locally finite, point-countable, resp.) open refinement. X is said to have a G_δ-*diagonal* if the diagonal $\Delta = \{(x, x) : x \in X\}$ is a G_δ-set in $X \times X$. X is called a *perfect space* if every open subset of X is an F_σ-set in X.

Definition 1.1. Let $\mathcal{P}$ be a base of a space X.

(1) $\mathcal{P}$ is a *uniform base* [1] (*uniform base at non-isolated points*, resp.) for X, if for each (non-isolated, resp.) point $x \in X$ and each countably infinite subset $\mathcal{P}'$ of $(\mathcal{P})_x$, $\mathcal{P}'$ is a neighborhood base at x.

(2) $\mathcal{P}$ is a *point-regular base* [1] (*point-regular base at non-isolated points*, resp.) for X if for each (non-isolated, resp.) point $x \in X$ and $x \in U$ with U open in X, $\{P \in (\mathcal{P})_x : P \not\subset U\}$ is finite.

In the definition, "at non-isolated points" means "at each non-isolated point of X." It is obvious that a uniform base (point-regular base, resp.) $\Rightarrow$ a uniform base at non-isolated points (point-regular base at non-isolated points, resp.), but we will see that a uniform base at non-isolated points (point-regular base at non-isolated points, resp.) $\not\Rightarrow$ a uniform base (point-regular base, resp.) by Example 4.1.

Definition 1.2. Let X be a space and $\{\mathcal{P}_n\}$ be a sequence of open subsets of X.

(1) $\{\mathcal{P}_n\}$ is called a *quasi-development* [8] for X if for every $x \in U$ with U open in X, there exists $n \in \mathbb{N}$ such that $x \in \mathrm{st}(x, \mathcal{P}_n) \subset U$.
(2) $\{\mathcal{P}_n\}$ is called a *development* (*development at non-isolated points*, resp.) for X if $\{\mathrm{st}(x, \mathcal{P}_n)\}_{n\in\mathbb{N}}$ is a neighborhood base at x in X for each (non-isolated, resp.) point $x \in X$.
(3) X is called *quasi-developable* (*developable*, *developable at non-isolated points*, resp.) if X has a quasi-development (development, development at non-isolated points, resp.).

It is obvious that every development for a space is a development at non-isolated points, but a space having a development at non-isolated points may not have a development; see Example 4.2.

Definition 1.3. Let $f : X \to Y$ be a map.

(1) f is a *compact map* (*s-map*, resp.) if each $f^{-1}(y)$ is compact (separable, resp.) in X;
(2) f is a *boundary-compact map* (*boundary-finite map*, *at most boundary-one map*, resp.) if each $\partial f^{-1}(y)$ is compact (finite, at most one point, resp.) in X;
(3) f is an *open map* if whenever U is open in X, then $f(U)$ is open in Y;
(4) f is a *bi-quotient map* (*countably bi-quotient map*, resp.) if for any $y \in Y$ and any (countable, resp.) family $\mathcal{U}$ of open subsets in X with $f^{-1}(y) \subset \cup\mathcal{U}$, there exists a finite subset $\mathcal{U}' \subset \mathcal{U}$ such that $y \in \mathrm{Int} f(\cup\mathcal{U}')$;
(5) f is a *pseudo-open map* if whenever $f^{-1}(y) \subset U$ with U open in X, then $y \in \mathrm{Int}(f(U))$.

It is easy to see that open $\Rightarrow$ bi-quotient $\Rightarrow$ countably bi-quotient $\Rightarrow$ pseudo-open $\Rightarrow$ quotient.

Definition 1.4. Let X be a space.

(1) A collection $\mathcal{U}$ of subsets of X is said to be Q (i.e., *interior-preserving*) if $\mathrm{Int}(\cap\mathcal{W}) = \cap\{\mathrm{Int} W : W \in \mathcal{W}\}$ for every $\mathcal{W} \subset \mathcal{U}$.

(2) An *ortho-base* [17] $\mathcal{B}$ for X is a base of X such that either $\cap\mathcal{A}$ is open in X or $\cap\mathcal{A} = \{x\} \notin \mathcal{I}(X)$ and $\mathcal{A}$ is a neighborhood base at x in X for each $\mathcal{A} \subset \mathcal{B}$. A space X is a *proto-metrizable space* [13] if it is a paracompact space with an ortho-base.

(3) A *sharp base* [2] $\mathcal{B}$ of X is a base of X such that, for every injective sequence $\{B_n\} \subset \mathcal{B}$, if $x \in \bigcap_{n\in\mathbb{N}} B_n$, then $\{\bigcap_{i\leq n} B_i\}_{n\in\mathbb{N}}$ is a neighborhood base at x.

(4) A base $\mathcal{B}$ of X is said to be a base of countable order (BCO) if, for any $x \in X$, if $\{B_i\} \subset \mathcal{B}$ is a strictly decreasing sequence, then $\{B_i\}_{i\in\mathbb{N}}$ is a neighborhood base at x.

It is well known ([2], [5], [6]) that

(1) uniform base $\Rightarrow$ σ-point-finite base $\Rightarrow$ σ-Q base;

(2) uniform base $\Rightarrow$ sharp base, developable space $\Rightarrow$ BCO, G_δ-diagonal;

(3) sharp base $\Rightarrow$ point-countable base.

Readers may refer to [11] and [18] for unstated definitions and terminology.

2. Some lemmas

In this section, some technical lemmas are given.

Lemma 2.1. *Let $\mathcal{P}$ be a base for a space X. Then the following are equivalent.*

(1) *$\mathcal{P}$ is a uniform base at non-isolated points for X;*

(2) *$\mathcal{P}$ is a point-regular base at non-isolated points for X.*

Proof: (2) $\Rightarrow$ (1) is trivial. We need only to prove (1) $\Rightarrow$ (2).

Let $\mathcal{P}$ be a uniform base at non-isolated points for X. If there exist a non-isolated point $x \in X$ and an open subset U in X with $x \in U$ such that $\{P \in (\mathcal{P})_x : P \not\subset U\}$ is infinite, take $\{P_n : n \in \mathbb{N}\} \subset \{P \in (\mathcal{P})_x : P \not\subset U\}$, and choose $x_n \in P_n \setminus U$ for each $n \in \mathbb{N}$. Then $\{P_n\}_{n\in\mathbb{N}}$ is a neighborhood base at x; thus, the sequence $\{x_n\}$ converges to x in X. Hence, $x_m \in U$ for some $m \in \mathbb{N}$, a contradiction. Therefore, $\mathcal{P}$ is a point-regular base at non-isolated points for X. □

Lemma 2.2. *Let $\{\mathcal{P}_n\}$ be a development at non-isolated points for a space X. If $\mathcal{P}_n$ is point-finite at each non-isolated point and $\mathcal{P}_{n+1}$*

refines $\mathcal{P}_n$ *for each* $n \in \mathbb{N}$, *then* $\mathcal{P} = \mathcal{I}(X) \cup (\bigcup_{n\in\mathbb{N}} \mathcal{P}_n)$ *is a uniform base at non-isolated points for* X.

Proof: Let x be a non-isolated point in X and $\{P_m : m \in \mathbb{N}\}$ be an infinite subset of $(\mathcal{P})_x$. By the point-finiteness, there exists $P_{m_k} \in \mathcal{P}_{n_k}$ such that $m_k < m_{k+1}$ and $n_k < n_{k+1}$ for each $k \in \mathbb{N}$. Since $\{\mathcal{P}_n\}$ is a development at non-isolated points for X, $\{P_{m_k}\}_{k\in\mathbb{N}}$ is a neighborhood base at x in X, so $\{P_m\}_{m\in\mathbb{N}}$ is a neighborhood base at x. Thus, $\mathcal{P}$ is a uniform base at non-isolated points for X. $\square$

Let $\mathcal{P}$ be a family of subsets of a space X. $\mathcal{P}$ is called *point-finite at non-isolated points* (*point-countable at non-isolated points*, resp.) if for each non-isolated point $x \in X$, x belongs to at most finite (countable, resp.) elements of $\mathcal{P}$. Let $\{\mathcal{P}_n\}$ be a development (development at non-isolated points, resp.) for X. $\{\mathcal{P}_n\}$ is said to be a *point-finite development* (*point-finite development at non-isolated points*, resp.) for X if each $\mathcal{P}_n$ is point-finite at each (non-isolated, resp.) point of X.

Lemma 2.3. *A space* X *has a uniform base at non-isolated points if and only if* X *has a point-finite development at non-isolated points.*

Proof: Sufficiency. It is easy to see by Lemma 2.2.

Necessity. Let $\mathcal{P}$ be a uniform base at non-isolated points for X. Then $\mathcal{P}$ is a point-regular base at non-isolated points by Lemma 2.1. We can assume that if $P \in \mathcal{P}$ and $P \subset I(X)$, P is a single point set.

Claim. Let x be a non-isolated point of X and $y \neq x$. Then $\{H \in \mathcal{P} : \{x, y\} \subset H\}$ is finite.

In fact, $\{H \in \mathcal{P} : \{x, y\} \subset H\} \subset (\mathcal{P})_x$. If $\{H \in \mathcal{P} : \{x, y\} \subset H\}$ is infinite, then it is a local base at x; hence, $y \to x$, a contradiction.

(a) $\mathcal{P}$ is point-countable at non-isolated points in X.

Let $x \in X$ be a non-isolated point. There is a non-trivial sequence $\{x_n\}$ converging to x. By the Claim, $\{P \in (\mathcal{P})_x : x_n \in P\}$ is finite for each n, then $(\mathcal{P})_x = \bigcup_{n\in\mathbb{N}}\{P \in (\mathcal{P})_x : x_n \in P\}$ is countable.

A family $\mathcal{F}$ of subsets of X is said to have the property ($\sharp$) if for any $F \in \mathcal{F} \setminus \mathcal{I}(X)$, then $\{H \in \mathcal{F} : F \subset H\}$ is finite.

(b) $\mathcal{P}$ has the property ($\sharp$).

Since $F \in \mathcal{P} \setminus \mathcal{I}(X)$, then F contains a non-isolated point and $|F| > 1$. By the Claim, $\mathcal{P}$ has the property ($\sharp$).

Put

$$\mathcal{P}^m = \{H \in \mathcal{P} : \text{if } H \subset P \in \mathcal{P}, \text{then } P = H\} \cup \mathcal{I}(X) \text{ and,}$$
$$\mathcal{P}' = (\mathcal{P} \setminus \mathcal{P}^m) \cup \mathcal{I}(X).$$

(c) $\mathcal{P}^m$ is an open cover and is point-finite at non-isolated points for X.

There exists $H_P \in \mathcal{P}^m$ such that $P \subset H_P$ for each $P \in \mathcal{P} \setminus \mathcal{I}(X)$ by (b). Thus, $\mathcal{P}^m$ is an open cover of X. If $\mathcal{P}^m$ is not point-finite at some non-isolated point $x \in X$, then there exists an infinite subset $\{H_n : n \in \mathbb{N}\}$ of $(\mathcal{P}^m)_x$. For each $n \in \mathbb{N}$, $H_n \not\subset H_1$, there exists $x_n \in H_{n+1} \setminus H_1$. Then $x_n \to x \in H_1$, a contradiction.

(d) $\mathcal{P}'$ is a point-regular base at non-isolated points for X.

Let $x \in U \setminus I(X)$ with U open in X. There exist $V, W \in \mathcal{P}$ and $y \in V \setminus \{x\}$ such that $x \in W \subset V \setminus \{y\} \subset V \subset U$. Thus, $W \in \mathcal{P}'$. Then $\mathcal{P}'$ is a base for X, and it is a point-regular base at non-isolated points for X.

Put $\mathcal{P}_1 = \mathcal{P}^m$ and $\mathcal{P}_{n+1} = [(\mathcal{P} \setminus \bigcup_{i \le n} \mathcal{P}_i) \bigcup \mathcal{I}(X)]^m$ for any $n \in \mathbb{N}$. Then $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ by (b).

(e) $\{\mathcal{P}_n\}$ is a point-finite development at non-isolated points for X.

Each $\mathcal{P}_n$ is point-finite at non-isolated points by (c) and (d). If $x \in U \setminus I(X)$ with U open in X, then $\{P \in (\mathcal{P})_x : P \not\subset U\}$ is finite; thus, there is $n \in \mathbb{N}$ such that $P \subset U$ whenever $x \in P \in \mathcal{P}_n$, i.e., $\mathrm{st}(x, \mathcal{P}_n) \subset U$. So $\{\mathcal{P}_n\}$ is a development at non-isolated points. $\square$

Lemma 2.4 ([3], [4], [14]). *The following are equivalent for a space X.*

(1) *X is an open compact image of a metric space;*
(2) *X is a pseudo-open compact image of a metric space;*
(3) *X has a uniform base;*
(4) *X has a point-regular base;*
(5) *X is a metacompact and developable space;*
(6) *X is a space with a point-finite development.*

Lemma 2.5. *Each pseudo-open, boundary-compact map is a bi-quotient map.*

Proof: Let $f : X \to Y$ be a pseudo-open, boundary-compact map. For each $y \in Y$ and a family $\mathcal{U}$ of open subsets in X with $f^{-1}(y) \subset \cup\mathcal{U}$, $\partial f^{-1}(y) \subset \cup\mathcal{U}'$ for some finite $\mathcal{U}' \subset \mathcal{U}$. We can assume that there exists $U \in \mathcal{U}'$ such that $U \cap f^{-1}(y) \neq \emptyset$. Thus, $y \in f(U)$. Let $V = (\cup\mathcal{U}') \cup \mathrm{Int}(f^{-1}(y))$. Then $f^{-1}(y) \subset V$. Since f is pseudo-open,

$$y \in \mathrm{Int}(f(V)) \subset f((\cup\mathcal{U}') \cup f^{-1}(y)) = f(\cup\mathcal{U}') \cup \{y\} = f(\cup\mathcal{U}'),$$

so $f(\cup\mathcal{U}')$ is a neighborhood of y in Y. Hence, f is a bi-quotient map. □

3. Main results

In this section, spaces with a uniform base at non-isolated points are discussed and some characterizations of images of metric spaces by boundary-compact maps are given.

Theorem 3.1. *The following are equivalent for a space X.*

1. *X is an open, boundary-compact image of a metric space;*
2. *X has a uniform base at non-isolated points;*
3. *X has a point-regular base at non-isolated points;*
4. *X has a point-finite development at non-isolated points.*

Proof: It is obvious that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) by Lemma 2.1 and Lemma 2.3.

(1) $\Rightarrow$ (4). Let M be a metric space and $f : M \to X$ be an open, boundary-compact map. By [11, 5.4.E], we can choose a sequence $\{\mathcal{B}_i\}$ of open covers of M such that $\{\mathrm{st}(K, \mathcal{B}_i)\}_{i\in\mathbb{N}}$ is a neighborhood base of K in M for each compact subset $K \subset M$. For each $i \in \mathbb{N}$, we can assume that $\mathcal{B}_{i+1}$ is a locally finite open refinement of $\mathcal{B}_i$, and set $\mathcal{P}_i = f(\mathcal{B}_i) \cup \mathcal{I}(X)$. Then $\mathcal{P}_i$ is an open cover of X for each $i \in \mathbb{N}$. If x is an accumulation point of X, then $\mathrm{Int} f^{-1}(x) = \emptyset$; thus, $f^{-1}(y) = \partial f^{-1}(x)$ is compact in M. Hence, $\{B \in \mathcal{B}_i : B \cap f^{-1}(x) \neq \emptyset\}$ is finite by the local finiteness of $\mathcal{B}_i$, i.e., $(\mathcal{P}_i)_x$ is finite. This shows that $\mathcal{P}_i$ is point-finite at non-isolated points. Next, we will prove that $\{\mathcal{P}_i\}$ is a development at non-isolated points for X. Let $x \in U \setminus I(X)$ with U open in X. Since $f^{-1}(x)$ is compact, there exists $m \in \mathbb{N}$ such that $\mathrm{st}(f^{-1}(x), \mathcal{B}_m) \subset f^{-1}(U)$, so $\mathrm{st}(x, \mathcal{P}_m) = \mathrm{st}(x, f(\mathcal{B}_m)) \subset U$. Thus, $\{\mathcal{P}_i\}$ is a point-finite development at non-isolated points for X.

(4) $\Rightarrow$ (1). First, a metric space M and a function $f : M \to X$ are defined: Let $\{\mathcal{P}_n\}$ be a point-finite development at non-isolated points for X. For each $n \in \mathbb{N}$, assume that $\mathcal{I}(X) \subset \mathcal{P}_n$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$, and endow Λ_n with the discrete topology. Put

$$M = \{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} \Lambda_n : \{P_{\alpha_n}\}_{n\in\mathbb{N}} \text{ is a neighborhood base at some } x_\alpha \in X\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} \Lambda_n$, is a metric space. Define a function $f : M \to X$ by $f((\alpha_n)) = x_\alpha$. Then $f((\alpha_n)) = \bigcap_{n\in\mathbb{N}} P_{\alpha_n}$, and f is well defined. $(f, M, X, \mathcal{P}_n)$ is called a Ponomarev system. It is easy to see that f is a map. The following will prove that f is an open boundary-compact map.

(a) f is an open map.

For any $\alpha = (\alpha_n) \in M, n \in \mathbb{N}$, put

$$B(\alpha_1, \alpha_2, \cdots, \alpha_n) = \{(\beta_i) \in M : \beta_i = \alpha_i \text{ whenever } i \leq n\}.$$

Then $f(B(\alpha_1, \alpha_2, \cdots, \alpha_n)) = \bigcap_{i\leq n} P_{\alpha_i}$. In fact, if $\beta = (\beta_i) \in B(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $f(\beta) = \bigcap_{i\in\mathbb{N}} P_{\beta_i} \subset \bigcap_{i\leq n} P_{\alpha_i}$. Thus

$$f(B(\alpha_1, \alpha_2, \cdots, \alpha_n)) \subset \bigcap_{i\leq n} P_{\alpha_i}.$$

On the other hand, let $x \in \bigcap_{i\leq n} P_{\alpha_i}$. Choose a countable family $\{P_{\beta_i}\}_{i\in\mathbb{N}}$ of subsets of X such that

(i) $x \in P_{\beta_i} \in \mathcal{P}_i$ for each $i \in \mathbb{N}$,
(ii) $\beta_i = \alpha_i$ whenever $i \leq n$, and
(iii) $P_{\beta_i} = \{x\}$ whenever $i > n$ and $x \in I(X)$.

Put $\beta = (\beta_i)$. Then $\beta \in B(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $f(\beta) = x$. Thus, $\bigcap_{i\leq n} P_{\alpha_i} \subset f(B(\alpha_1, \alpha_2, \cdots, \alpha_n))$.

In conclusion, $f(B(\alpha_1, \alpha_2, \cdots, \alpha_n)) = \bigcap_{i\leq n} P_{\alpha_i}$. Since

$$\{B(\alpha_1, \alpha_2, \cdots, \alpha_n) : (\alpha_i) \in M, n \in \mathbb{N}\}$$

is a base of M, f is an open map.

(b) f is a boundary-compact map.

Let $x \in X$. If $x \in I(X)$, then $\partial f^{-1}(x) = \emptyset$. If $x \notin I(X)$, $\partial f^{-1}(x) = f^{-1}(x)$ by (b). For each $i \in \mathbb{N}$, let $\Gamma_i = \{\alpha \in \Lambda_i : x \in P_\alpha\}$. Then Γ_i is finite. Thus, $\prod_{i\in\mathbb{N}} \Gamma_i$ is a compact subset of $\prod_{i\in\mathbb{N}} \Lambda_i$. We need only to prove $f^{-1}(x) = \prod_{i\in\mathbb{N}} \Gamma_i$. Indeed, if

$\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Gamma_i$, then $\{P_{\alpha_i}\}_{i\in\mathbb{N}}$ is a neighborhood base at x for X. Thus, $\alpha \in M$ and $f(\alpha) = x$, so $\prod_{i\in\mathbb{N}} \Gamma_i \subset f^{-1}(x)$. On the other hand, if $\alpha = (\alpha_i) \in f^{-1}(x)$, then $x \in \bigcap_{i\in\mathbb{N}} P_{\alpha_i}$ and $\alpha \in \prod_{i\in\mathbb{N}} \Gamma_i$. So $f^{-1}(x) \subset \prod_{i\in\mathbb{N}} \Gamma_i$. Thus, $\partial f^{-1}(x) = f^{-1}(x) = \prod_{i\in\mathbb{N}} \Gamma_i$ is compact. □

In the Ponomarev system $(f, M, X, \mathcal{P}_n)$, it always holds that $f^{-1}(x) \subset \prod_{i\in\mathbb{N}}\{\alpha \in \Lambda_i : x \in P_\alpha\}$ for each $x \in X$. The following corollary is obtained.

Corollary 3.2. *A space X has a point-countable base which is uniform at non-isolated points if and only if X is an open boundary-compact, s-image of a metric space.*

Corollary 3.3. *Each space having a uniform base at non-isolated points is preserved by an open, boundary-finite map.*

Proof: Let $f : X \to Y$ be an open boundary-finite map where X has a uniform base at non-isolated points. There exist a metric space M and an open boundary-compact map $g : M \to X$ by Theorem 3.1. Since $\partial(f \circ g)^{-1}(y) \subset \bigcup\{\partial g^{-1}(x) : x \in \partial f^{-1}(y)\}$ for each $y \in Y$, $f \circ g : M \to Y$ is an open boundary-compact map. Hence, Y has an uniform base at non-isolated points. □

Theorem 3.4. *Let X be a space having a uniform base at non-isolated points. Then*

(1) *X is a quasi-developable space;*
(2) *X has an ortho-base and a σ-Q base.*

Proof: By Theorem 3.1, let $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ be a point-finite development at non-isolated points for X. Put $\mathcal{P}_0 = \mathcal{I}(X)$. It is easy to check that $\{\mathcal{P}_n\}_{n\in\omega}$ is a quasi-development for X.

Let $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$. Then $\mathcal{P}$ is a σ-Q base and an ortho-base for X.

First, $\mathcal{P}_n$ is interior-preserving for each $n \in \mathbb{N}$. Indeed, for each $\mathcal{A} \subset \mathcal{P}_n$, if $x \in \cap\mathcal{A} - I(X)$, then $(\mathcal{P}_n)_x$ is finite; thus, $\cap\mathcal{A}$ is a neighborhood of x in X. So $\mathcal{P}$ is a σ-Q base for X.

Secondly, let $\mathcal{A} \subset \mathcal{P}$ with $\cap\mathcal{A}$ not open in X. Then there exists $x \in \cap\mathcal{A}$ such that $\cap\mathcal{A}$ is not a neighborhood of x in X; thus, x is a non-isolated point and $(\mathcal{P}_n)_x$ is finite for each $n \in \mathbb{N}$. Let $x \in U$ with U open in X. There exists $n \in \mathbb{N}$ such that $x \in \mathrm{st}(x, \mathcal{P}_n) \subset U$. Choose $m \geq n$ and $A \in \mathcal{A} \cap \mathcal{P}_m$. Then $A \subset \mathrm{st}(x, \mathcal{P}_n) \subset U$; thus, $\mathcal{A}$

is a neighborhood base at x in X. So $\cap\mathcal{A}$ is a single point subset. Hence, $\mathcal{P}$ is an ortho-base for X. □

Corollary 3.5. *Let X be a space having a uniform base at non-isolated points. Then (1) ⇒ (2) ⇔ (3) in the following.*

(1) *X has a sharp base;*
(2) *X is a developable space;*
(3) *$I(X)$ is an F_σ-set in X.*

Proof: (1) ⇒ (3) is proved in [7, Theorem 3.1] for any space X. (2) ⇒ (3) is obvious because each open subset of a developable space is an F_σ-set.

To prove (3) ⇒ (2), let $\{\mathcal{B}_n\}$ be a point-finite development at non-isolated points for X by Theorem 3.1. Since $I(X)$ is an F_σ-set, there exists a sequence $\{G_n\}$ of open subsets of X such that $X - I(X) = \bigcap_{n\in\mathbb{N}} G_n$. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{G_n\} \cup \{\{x\} : x \in X - G_n\}.$$

Then $\{\mathcal{B}_n, \mathcal{U}_n\}$ is a development for X. Hence, X is a developable space. □

The following corollary holds by Lemma 2.4.

Corollary 3.6. *A space X is an open compact image of a metric space if and only if X is a perfect, metacompact space, which is an open boundary-compact image of a metric space.*

By the corollary, some metrizable theorems on spaces with a uniform base at non-isolated points can be obtained. For example, let X be a space with a uniform base at non-isolated points, then X is metrizable if and only if it is a perfect, collectionwise normal space.

Now, a special space with a uniform base at non-isolated points is discussed. Let (X, τ) be a space and $A \subset X$. X is said to be discretizable by A if X is endowed with the topology generated by $\tau \cup \{\{x\} : x \in A\}$ as a base for X [17]. Denote the discretizable space of X by X_A.

It is obvious that the topology of a space X is coarser than the discretizable topology of X_A. If X has a uniform base, then X_A not only has a G_δ-diagonal and a uniform base at non-isolated points, but also has a σ-point finite base. In [13, Theorem 3.1], Gary Gruenhage and Phillip Zenor have shown that a space is a discretization of a metric space if and only if it is a proto-metrizable space having a G_δ-diagonal.

Theorem 3.7. *Each discretizable space of a space having a uniform base is an open compact and at most boundary-one image of a space having a uniform base.*

Proof: Let X be a space having a uniform base. By Lemma 2.4, there is a point-finite development $\{\mathcal{U}_m\}$ for X, where $\mathcal{U}_{m+1}$ refines $\mathcal{U}_m$ for each $m \in \mathbb{N}$. For each $A \subset X$, put

$$H = (X \times \{0\}) \cup (A \times \mathbb{N});$$
$$V(x,m) = \{x\} \times (\{0\} \cup \{n \in \mathbb{N} : n \geq m\}), x \in X, m \in \mathbb{N};$$
$$W(J,m) = ((J \cap (X - A)) \times \{0\}) \cup ((J \cap A) \times \{n \in \mathbb{N} : n \geq m\}), J \subset X, m \in \mathbb{N}.$$

Endow H with a base consisting of the following elements:

$V(x,m), \forall x \in A, m \in \mathbb{N}$;
$W(J,m), \forall$ open subset $J \subset X, m \in \mathbb{N}$;
$\{x\}, x \in A \times \mathbb{N}$.

Then H is a T_2-space.

For any $m \in \mathbb{N}$, let

$$\mathcal{P}_m = \{V(x,m) : x \in A\} \cup \{W(U,m) : U \in \mathcal{U}_m\} \cup \{\{h\} : h \in A \times \{1, 2, \cdots, m-1\}\}.$$

Then $\{\mathcal{P}_m\}_{m \geq 2}$ is a point-finite development for H. Hence, H has a uniform base.

Let $\pi_1|_H : H \to X_A$ be the projective map. It is easy to see that $\pi_1|_H$ is an open compact and at most boundary-one map. □

Hence, each discretizable space of a space having a uniform base is in MOBI [8].

Liu [16] gave some characterizations of quotient (pseudo-open, resp.) boundary-compact images of metric spaces. The following are further results.

Theorem 3.8. *The following are equivalent for a space X.*

(1) *X is first-countable;*
(2) *X is an image of a metric space under a pseudo-open, at most boundary-one (boundary-compact, resp.) map;*
(3) *X is an image of a metric space under a bi-quotient, at most boundary-one (boundary-compact, resp.) map.*

Proof: (1) $\Leftrightarrow$ (2) was proved in [16, Corollary 2.1], and (2) $\Leftrightarrow$ (3) is true by Lemma 2.5. □

Theorem 3.9. *The following are equivalent for a space X.*

(1) *X has a point-countable base;*
(2) *X is a countably bi-quotient, s-image of a metric space;*
(3) *X is a pseudo-open, boundary-compact and s-image of a metric space;*
(4) *X is a bi-quotient, at most boundary-one and countable-to-one image of a metric space.*

Proof: Liu proved in [16] that a space has a point-countable base if and only if it is a pseudo-open, at most boundary-one and countable-to-one image of a metric space. Thus, (1) $\Leftrightarrow$ (4) by Lemma 2.5. (4) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (2) by Lemma 2.5, and (2) $\Leftrightarrow$ (1) by [21]. $\square$

4. Examples

In this section, we provide some examples which show certain relationships among boundary-compact images of metric spaces and generalized metric spaces.

Example 4.1. Let X be the closed unit interval $\mathbb{I} = [0, 1]$ and B be a Bernstein subset of X. In other words, B is an uncountable set which contains no uncountable closed subset of X. The discretizable space X_B is called the *Michael line* [20].

Let X^* be a copy of X_B and $f : X_B \to X^*$ be a homeomorphism. Put $Z = X_B \bigoplus X^*$, and let Y be a quotient space obtained from Z by identifying $\{x, f(x)\}$ to a point for each $x \in X_B \setminus B$. Then

(1) X_B is a discretizable space of the metric space $\mathbb{I}$, so, by Theorem 3.7, it is a proto-metrizable space and an open compact, at most boundary-one image of a space with a uniform base.
(2) X_B is not a BCO space; hence, it is not an open compact image of a metric space;
(3) Y is an open boundary-compact, s-image of a metric space;
(4) Y has no G_δ-diagonal by [23, Example 1].

It is obvious that X_B is a paracompact space which is a discretizable space of the metric space $\mathbb{I}$. If X_B is BCO, it is a developable space, and then B is an F_σ-set in X_B, a contradiction. Thus, X_B is not BCO.

It is easy to check that Y has a point-countable base which is uniform at non-isolated points. Hence, Y is an open boundary-compact, s-image of a metric space by Corollary 3.2.

Example 4.2. Let $\psi(D)$ be the *Isbell-Mrówka* space [22], here $|D| \geq \aleph_0$. Then

(1) $\psi(D)$ is an open, boundary-compact image of a metric space;
(2) $\psi(D)$ is not a metalindelöf space;
(3) $\psi(D)$ is a developable space if $|D| = \aleph_0$;
(4) $\psi(D)$ is not a perfect space if $|D| \geq \mathbf{c}$.

A collection $\mathcal{C}$ of subsets of an infinite set D is said to be *almost disjoint* if $A \cap B$ is finite whenever $A \neq B \in \mathcal{C}$. Let $\mathcal{A}$ be an almost disjoint collection of countably infinite subsets of D and maximal with respect to the properties. Then $|\mathcal{A}| \geq |D|^+$ [15]. The Isbell-Mrówka space $\psi(D)$ is the set $\mathcal{A} \cup D$ endowed with the following topology: The points of D are isolated. Basic neighborhoods of a point $A \in \mathcal{A}$ are the sets of the form $\{A\} \cup (A - F)$ where F is a finite subset of D.

Let $X = \psi(D), \mathcal{A} = \{A_\alpha\}_{\alpha\in\Lambda}$, and each $A_\alpha = \{x(\alpha, n) : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, put

$$\mathcal{B}_n = \{\{A_\alpha\} \cup \{x(\alpha, m) : m \geq n\} : \alpha \in \Lambda\} \cup \{\{x\} : x \in D\}.$$

It is easy to see that $\{\mathcal{B}_n\}$ is a point-finite development for X. Thus, X is the open, boundary-compact image of a metric space by Theorem 3.1. Since an open cover $\{\{A_\alpha\} \cup D\}_{\alpha\in\Lambda}$ of X has no point-countable open refinement, X is not a metalindelöf space. Thus, X is not an open s-image of a metric space, and X is not a discretizable space of a space with a uniform base by Theorem 3.7.

If D is countable, it is obvious that $\psi(D)$ is a developable space. Hence, $\psi(D)$ has a G_δ-diagonal, but $\psi(D)$ has no point-countable base because $\psi(D)$ is not a metalindelöf space.

If $|D| \geq \mathbf{c}$, $\psi(D)$ is not a developable space [9]; thus, $\psi(D)$ is not perfect by Corollary 3.5.

Example 4.3. There is a space X such that

(1) X has a sharp base;
(2) X does not have a uniform base at non-isolated points;
(3) X is an open compact and countable-to-one image of a space with a uniform base.

A space X having properties (1)–(3) is constructed in [2, Example 5.1], where it is shown that X has a non-developable space with a sharp base. Since X has no isolated point, it is not an open, boundary-compact image of a metric space and does not have a uniform base at non-isolated points by Theorem 3.1. J. Chaber, in [10, Example 4.5], proved that X is an open compact and countable-to-one image of a space with a uniform base.

Example 4.4. There is a bi-quotient, at most boundary-one image X of a metric space such that X is neither a pseudo-open s-image of a metric space, nor an open, boundary-compact image of a metric space.

Let $X = \mathbb{R}^2$ be endowed with the butterfly topology [19]. It is easy to see that X is a first-countable, paracompact space without any isolated point. Since X is a first-countable space, then X is a bi-quotient, at most boundary-one image of a metric space by Theorem 3.8. Since X does not have a point-countable base [18, Example 1.8.3], X is not a countably bi-quotient s-image of a metric space by Theorem 3.9. Because each pseudo-open map from a space onto a first-countable space is countably bi-quotient [21], X is not a pseudo-open s-image of a metric space. If X is an open, boundary-compact image of a metric space, X is an open compact image of a metric space, for X does not contain any isolated point. So X is a developable space by Lemma 2.4. Thus, X is a metric space, a contradiction.

Example 4.5. There is a proto-metrizable space without any uniform base at non-isolated points.

Gruenhage in [12, p. 363] constructed a proto-metrizable X which is not a γ-space. Hence, X has no σ-Q base by [18, Proposition 1.7.10], and it has no uniform base at non-isolated points by Theorem 3.4.

Example 4.6. There is a space such that it is an open compact image of a metric space, which is not any open, at most boundary-one image of a metric space.

Yoshio Tanaka in [24, Example 3.1] constructed a non-regular T_2-space X which is an open, at most two-to-one image of a metric space. Since X has no isolated point, it is not an open, at most boundary-one image of a metric space. Otherwise, X is an image of a metric space under an open and bijective map, and then X is homeomorphic to a metric space, a contradiction.

5. Questions

Some questions are posed in this final section.

Question 5.1. Let a space X have a point-countable base. If X has a uniform base at non-isolated points, is X an open, boundary-compact, s-image of a metric space?

Question 5.2. Is an open and boundary-compact s-image of a metric space an open, boundary-compact and countable-to-one image of a metric space?

Question 5.3. How could a discretizable space of a space with a uniform base be characterized by a certain image of a metric space? For example, is the open compact and at most boundary-one image of a space with a uniform base a discretizable space of a space with a uniform base?

Question 5.4. How could a space which is an open, at most boundary-one, s-image of a metric space be characterized?

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