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A NOTE ON $\Sigma^{\#}$-SPACES

SHOU LIN AND RONGXIN SHEN

Abstract. A space is called a $\Sigma^{\#}$-space if it has a σ-closure-preserving quasi-(mod k)-network. In this paper, a new characterization of $\Sigma^{\#}$-spaces is given, which answers a question posed by the first author. Also we characterize spaces with σ-cushioned quasi-(mod k)-networks by means of g-functions.

1. Introduction

The class of Σ-spaces, which was defined by Keiô Nagami [8], plays an important role in the theory of generalized metric spaces. E. Michael [7] and Akihiro Okuyama [10] introduced $\Sigma^{\#}$-spaces and Σ^*-spaces, respectively, as some generalizations, and additionally, Okuyama proved the following theorem.

Theorem 1.1 ([10, Lemma 3.5]). *If X is a Σ^*-space (a $\Sigma^{\#}$-space, resp.), then X has a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of hereditarily closure-preserving (closure-preserving, resp.) closed covers of X such that any sequence $\{x_n : n \in \mathbb{N}\}$ with $x_n \in C(x, \mathcal{F}_n)$ for some $x \in X$ has a cluster point. In particular, X is a Σ-space if and only if X has a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of locally finite closed covers of X such that any sequence $\{x_n : n \in \mathbb{N}\}$ with $x_n \in C(x, \mathcal{F}_n)$ for some $x \in X$ has a cluster point.*

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It is natural to consider whether the inverse is true for Σ^*-spaces and $\Sigma^\#$-spaces in the above theorem. So the first author raised the following question.

Question 1.2 ([4, Question 3.2.4])**.** If X has a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of closure-preserving closed covers of X such that any sequence $\{x_n : n \in \mathbb{N}\}$ with $x_n \in C(x, \mathcal{F}_n)$ for some $x \in X$ has a cluster point, is X a $\Sigma^\#$-space?

In this paper, we give an affirmative answer to this question. We also discuss the class of spaces with σ-cushioned quasi-(mod k)-networks.

Let $\mathcal{P} = \{P_\alpha : \alpha \in I\}$ be a collection of subsets of a topological space (X, τ). $\mathcal{P}$ is called *closure preserving* [5] if

$$\overline{\cup\{P_\alpha : \alpha \in J\}} = \cup\{\overline{P_\alpha} : \alpha \in J\}$$

for any $J \subset I$; $\mathcal{P}$ is called *hereditarily closure preserving* [2] if any $\{B_\alpha : \alpha \in I\}$ with $B_\alpha \subset P_\alpha$ for each $\alpha \in I$ is closure-preserving. Throughout this paper, all spaces are T_1; we denote $\mathbb{N}$ by the natural numbers and $C(x, \mathcal{P})$ by $\cap\{P \in \mathcal{P} : x \in P\}$ for each $x \in X$.

2. Main results

Definition 2.1. A cover $\mathcal{P}$ of a space X is called a *quasi-(mod k)-network* [10] if there is a closed cover $\mathcal{H}$ of X by countably compact subsets such that whenever $H \subset U$ with $H \in \mathcal{H}$ and U is open in X, then $H \subset P \subset U$ for some $P \in \mathcal{P}$. X is called a Σ-*space* [8] (a Σ^*-*space* [10], a $\Sigma^\#$-*space* [7], resp.) if it has a σ-locally finite (σ-hereditarily closure-preserving, σ-closure-preserving, resp.) closed quasi-(mod k)-network.

Theorem 2.2. *For a space X, if X has a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of closure-preserving closed covers of X such that any sequence $\{x_n : n \in \mathbb{N}\}$ with $x_n \in C(x, \mathcal{F}_n)$ for some $x \in X$ has a cluster point, then X is a $\Sigma^\#$-space.*

Proof: Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of closure-preserving closed covers of X such that any sequence $\{x_n : n \in \mathbb{N}\}$ with $x_n \in C(x, \mathcal{F}_n)$ for some $x \in X$ has a cluster point. Without loss of generality, we can assume that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, put

$$\mathcal{P}_n = \{\cap\mathcal{E} : \emptyset \neq \mathcal{E} \subset \mathcal{F}_n\}.$$

Then each $\mathcal{P}_n$ is a closure-preserving closed cover of X. In fact, it is easy to see each $\mathcal{P}_n$ is a closed cover of X. Let $\{\cap\mathcal{E}_\alpha : \alpha \in \Lambda\} \subset \mathcal{P}_n$, where $\mathcal{E}_\alpha \subset \mathcal{F}_n$ for each $\alpha \in \Lambda$. If $x \notin \cup\{\cap\mathcal{E}_\alpha : \alpha \in \Lambda\}$, then there is an $E_\alpha \in \mathcal{E}_\alpha$ such that $x \notin E_\alpha$ for each $\alpha \in \Lambda$. Since $\mathcal{F}_n$ is closure-preserving,

$$\begin{aligned} x \in X - \cup\{E_\alpha : \alpha \in \Lambda\} &= X - \overline{\cup\{E_\alpha : \alpha \in \Lambda\}} \\ &\subset X - \overline{\cup\{\cap\mathcal{E}_\alpha : \alpha \in \Lambda\}}, \end{aligned}$$

which shows that $\mathcal{P}_n$ is closure-preserving. Moreover, we have $C(x,\mathcal{P}_n) \subset C(x,\mathcal{F}_n)$ and $C(x,\mathcal{P}_n) \in \mathcal{P}_n$ for each $x \in X$ and $n \in \mathbb{N}$.

Put

$$\mathcal{H} = \{\bigcap_{n\in\mathbb{N}} C(x,\mathcal{P}_n) : x \in X\}.$$

Then $\mathcal{H}$ is a closed cover of X by countably compact sets. For each $x \in X$ and an open set U with $\bigcap_{n\in\mathbb{N}} C(x,\mathcal{P}_n) \subset U$, we prove that there is $m \in \mathbb{N}$ such that $C(x,\mathcal{P}_m) \subset U$; thus, $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ is a quasi-(mod k)-network for X. Suppose not; we can choose a sequence $\{x_n : n \in \mathbb{N}\}$ with $x_n \in C(x,\mathcal{P}_n) - U$ for each $n \in \mathbb{N}$, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point y. Since $y \in \overline{\{x_m : m \geq n\}} \subset C(x,\mathcal{P}_n)$ for each $n \in \mathbb{N}$, $y \in \bigcap_{n\in\mathbb{N}} C(x,\mathcal{P}_n) \subset U$. However,

$$y \in \overline{\{x_n : n \in \mathbb{N}\}} \subset \overline{X - U} = X - U,$$

a contradiction. Hence, X is a $\Sigma^{\#}$-space. □

Theorem 2.2 gives an affirmative answer to Question 1.2. However, the following question is still open.

Question 2.3. If X has a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of hereditarily closure-preserving closed covers of X such that any sequence $\{x_n : n \in \mathbb{N}\}$ with $x_n \in C(x,\mathcal{F}_n)$ for some $x \in X$ has a cluster point, is X a Σ^*-space?

For a topological space (X,τ), a function $g : \mathbb{N}\times X \to \tau$ is called a *g-function* [1] if $x \in g(n+1,x) \subset g(n,x)$ for each $x \in X$ and $n \in \mathbb{N}$. $g(n,A)$ denotes $\bigcup_{x\in A} g(n,x)$ for $A \subset X$. Let $\mathcal{P}$ be a collection of pairs of subsets of X: $\mathcal{P}$ is called a *quasi-(mod k)-network* [3] for X if there is a closed cover $\mathcal{H}$ of countably compact subsets of X such that whenever $H \subset U$ with $H \in \mathcal{H}$ and U is open in X, then $H \subset P_1 \subset P_2 \subset U$ for some $(P_1,P_2) \in \mathcal{P}$; $\mathcal{P}$ is called *cushioned* [6] if

$$\overline{\cup\{P_1 : (P_1,P_2) \in \mathcal{P}'\}} \subset \cup\{P_2 : (P_1,P_2) \in \mathcal{P}'\}$$

for each $\mathcal{P}' \subset \mathcal{P}$.

Clearly, every $\Sigma^{\#}$-space is a space with a σ-cushioned quasi-(mod k)-network. In [9], Jun-iti Nagata gave the following characterization of $\Sigma^{\#}$-spaces.

Theorem 2.4 ([9]). *X is a $\Sigma^{\#}$-space if and only if X has a g-function satisfying*

(1) *for each $x, y \in X$ and $n \in \mathbb{N}$, if $x \in g(n, y)$, then $g(n, x) \subset g(n, y)$;*
(2) *for each $x \in X$ and $\{x_n : n \in \mathbb{N}\} \subset X$, if $x \in g(n, x_n)$ for each $n \in \mathbb{N}$, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point.*

Similarly, we characterize spaces with σ-cushioned quasi-(mod k)-networks by means of g-functions.

Theorem 2.5. *A space X has a σ-cushioned quasi-(mod k)-network if and only if there are a closed cover $\mathcal{H}$ of X by countably compact sets and a g-function such that for each $H \in \mathcal{H}$ and sequence $\{x_n : n \in \mathbb{N}\} \subset X$, if $H \cap g(n, x_n) \neq \emptyset$ for each $n \in \mathbb{N}$, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point in H.*

Proof: Necessity. Let $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ be a σ-cushioned quasi-(mod k)-network for X with respect to a closed cover $\mathcal{H}$ of countably compact sets, where each $\mathcal{P}_n$ is cushioned and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{N}$. Define $g : \mathbb{N} \times X \to \tau$ as

$$g(n, x) = X - \overline{\cup\{P_1 : (P_1, P_2) \in \mathcal{P}_n, x \notin P_2\}}.$$

It is easy to verify that g is a g-function for X. Let $H \in \mathcal{H}$ and $\{x_n : n \in \mathbb{N}\} \subset X$ with $H \cap g(n, x_n) \neq \emptyset$. If $\{x_n : n \in \mathbb{N}\}$ has no cluster point in H, then there is $m \in \mathbb{N}$ such that $H \cap \overline{\{x_n : n \geq m\}} = \emptyset$. Otherwise, for each $k \in \mathbb{N}$, put $F_k = \overline{\{x_n : n \geq k\}}$. Then $\{H \cap F_k\}$ is a decreasing sequence of nonempty closed subsets of H. Hence, there is $h \in \bigcap_{k \in \mathbb{N}} (H \cap F_k)$. This is a contradiction. So there is $m \in \mathbb{N}$ such that $H \subset X - \overline{\{x_n : n \geq m\}}$. Then

$$H \subset P_1 \subset P_2 \subset X - \overline{\{x_n : n \geq m\}}$$

for some $i \in \mathbb{N}$ and $(P_1, P_2) \in \mathcal{P}_i$. Pick $j \geq \max\{m, i\}$, then $g(j, x_j) \subset X - P_1 \subset X - H$, a contradiction. It implies that $\{x_n : n \in \mathbb{N}\}$ has a cluster point in H.

Sufficiency. For each $n \in \mathbb{N}$, put $P(n, U) = X - g(n, X - U)$ for each $U \in \tau$ and $\mathcal{P}_n = \{(P(n, U), U) : U \in \tau\}$, then $\mathcal{P}_n$ is a cushioned family. We need only to show that for each $H \in \mathcal{H}$ and $H \subset U \in \tau$, there is $m \in \mathbb{N}$ such that $H \subset P(m, U)$. In fact, if $H \not\subset P(n, U)$ for each $n \in \mathbb{N}$, then there is a sequence $\{x_n : n \in \mathbb{N}\} \subset X - U$ with $H \cap g(n, x_n) \neq \emptyset$ for each $n \in \mathbb{N}$, thus $\{x_n : n \in \mathbb{N}\}$ has a cluster point in $H - U$, a contradiction. Therefore, X has a σ-cushioned quasi-(mod k)-network. □

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