

COUNTEREXAMPLES ON THE IMAGES OF LOCALLY SEPARABLE METRIC SPACES

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ABSTRACT. In this paper, we give negative answers to several questions on the images of locally separable, metric spaces.

1. INTRODUCTION

All spaces considered here are assumed to be regular T_1 .

First, we recall some definitions.

The symbol \mathbb{N} is the set of positive integers.

A continuous map is called an *s-map* if each fiber of the map is separable.

A continuous map $f : X \rightarrow Y$ is called *compact-covering* if every compact $K \subset Y$ is the image of a compact $C \subset X$.

Let \mathcal{P} be a family of subsets of a space X . Then \mathcal{P} is called a *cs-network* [5] if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to a point $x \in X$ and any neighborhood U of x , there exist $P \in \mathcal{P}$ and $m \in \mathbb{N}$ such that $\{x, x_n : n \geq m\} \subset P \subset U$. \mathcal{P} is called a *cs*-network* [3] if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to a point $x \in X$ and any neighborhood U of x , there exist $P \in \mathcal{P}$ and a subsequence $\{x_{n_j}\}_{j \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that $\{x, x_{n_j} : j \in \mathbb{N}\} \subset P \subset U$. \mathcal{P} is called a *k-network* [15] if for any compact set $K \subset X$ and an open set U with $K \subset U$, there exists a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that

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$$K \subset \bigcup \mathcal{P}' \subset U.$$

A space is called an \aleph_0 -space [14] if it has a countable k -network.

In this paper, we give negative answers to the following questions.

Question 1.1. [9, Conjecture 5.1.3] Let X be a quotient s -image of a metric space. Is X a quotient s -image of a locally separable, metric space if each first countable subset of X is locally separable?

Question 1.2. [11, Question 3.4] Let X be a regular Fréchet space with a point-countable k -network. Is X a closed image of a locally separable, metric space if each first countable closed subspace of X is locally separable?

Question 1.3. [8, Question 6] Let X be a regular Fréchet space with a point-countable k -network. Is X a space with a point-countable k -network consisting of separable subsets of X if each first countable closed subspace of X is locally separable?

Question 1.4. [13, Question 1.2] Is it true that a space X is a closed s -image of a locally separable, metric space if and only if X is a regular Fréchet space with a point-countable cs^* -network, and each first countable closed subspace of X is locally separable?

Question 1.5. [12, Question 2.6(2)] Is a hereditarily separable, sequential space with a point-countable cs^* -network an \aleph_0 -space?

Question 1.6. [10, Question 3.6] Let X be a quotient s -image of a metric space. Is X a quotient s -image of a locally compact, metric space if each first countable closed subset of X is locally compact?

Since a space X is a closed image of a locally separable, metric space if and only if X is a Fréchet space with a point-countable k -network consisting of separable subsets of X [16, Corollary 3.6], Question 1.2 and Question 1.3 are equivalent.

2. COUNTEREXAMPLES

First, we answer Question 1.1 through Question 1.4. We recall the space Y constructed in [18, Example 2.3]. Let P be a Bernstein set of the closed unit interval $\mathbb{I} = [0, 1]$ [2, p. 339]. In other words, P is an uncountable set which contains no uncountable closed set of \mathbb{I} . Let X be the space obtained from \mathbb{I} by isolating the points of

P . The space X was considered in [4, Example 9.4]. Our space Y is the quotient space obtained from X by collapsing the set $X \setminus P$ to the one-point ∞ .

Proposition 2.1. [18, Example 2.3] *The space Y has the following properties.*

- (1) Y is regular T_1 and Fréchet;
- (2) Y has a point-countable closed family \mathcal{P} which is both a cs -network (hence, cs^* -network) and a k -network;
- (3) every first countable closed subset of Y is countable;
- (4) Y does not have any star-countable k -network.

We observe further properties of Y . By combining Proposition 2.1 with Theorem 2.2 below, we can easily see that Y is a counterexample for Question 1.1 through Question 1.4.

Theorem 2.2. *The space Y has the following properties, too.*

- (a) Y is a quotient s -image of a metric space;
- (b) every first countable subset of Y is a discrete space or a countable space; hence, every first countable subset of Y is locally separable;
- (c) Y does not have any point-countable cs^* -network of separable subsets; in particular, Y is not a quotient s -image of a locally separable, metric space;
- (d) Y is not a closed image of a locally separable, metric space;
- (e) Y does not have any point-countable k -network consisting of separable subsets.

Proof: (a) It is proved in [4, p. 315] that every k -space with a point-countable k -network of closed sets is a quotient s -image of a metric space.

(b) Let A be a first countable subset of Y . If $\infty \notin A$, then A is discrete. Assume $\infty \in A$. Then A is closed in Y ; hence, A is countable by Proposition 2.1(3).

(c) If Y has a point-countable cs^* -network of separable subsets, then Y has a point-countable k -network of separable subsets since each point-countable cs^* -network for a sequential space is a point-countable k -network [19, Corollary 1.5]. Hence, (e) implies (c).

It is known in [12, Theorem 2.2] that if a space is a quotient s -image of a locally separable, metric space, then it has a point-countable cs^* -network of separable subsets. Hence, Y is not a quotient s -image of a locally separable, metric space.

(d) It is known in [16, Corollary 3.6] that a space is a closed image of a locally separable, metric space if and only if it is a Fréchet space with a star-countable k -network. Apply Proposition 2.1(4).

(e) It is known in [16, Theorem 3.5] that every Fréchet space with a point-countable k -network consisting of separable subsets has a star-countable k -network. Apply Proposition 2.1(4). \square

We remark that the following two statements are incorrect in view of our example Y .

Statement 2.3. [12, Lemma 2.4] Suppose \mathcal{P} is a point-countable collection of subsets of X , which is closed under finite intersections. Let

$$\mathcal{H} = \{P \in \mathcal{P} : P \text{ is a hereditarily separable subspace of } X\}.$$

Then \mathcal{H} is a cs^* -network for X if and only if \mathcal{P} is a cs^* -network for X and every first countable subspace of X is locally separable.

Statement 2.4. [12, Corollary 2.5] The following conditions are equivalent for a space X :

- (i) X is a quotient s -image of a metric space and every first countable subspace of X is locally separable;
- (ii) X is a sequential space with a point-countable cs^* -network consisting of hereditarily separable subspaces.

Indeed, let \mathcal{P} be a point-countable cs^* -network for Y ; see Proposition 2.1(2). Consider the family $\mathcal{P}' = \{P_0 \cap \cdots \cap P_n : P_i \in \mathcal{P}, n \in \omega\}$. A point-countable cs^* -network is closed under finite intersections. But $\mathcal{H} = \{P \in \mathcal{P}' : P \text{ is hereditarily separable}\}$ is not a cs^* -network by Theorem 2.2 (c). Thus, Statement 2.3 is incorrect.

By Theorem 2.2(a) and (b), Y satisfies condition (i) in Statement 2.4. But Y does not satisfy condition (ii) in Statement 2.4 by Theorem 2.2(c). Thus, Statement 2.4 is also incorrect.

Next, we answer Question 1.5. In [20, Example 1.6(2)], a space X_A is constructed as follows. Let A be an uncountable subset of the

closed unit interval \mathbb{I} . For each $x \in A$, let $S_x = \{(x, 0)\} \cup \{(x, 1/n) : n \in \mathbb{N}\}$, and let

$$\mathcal{C} = \{S_x : x \in A\} \cup \{A \times \{0\}\} \cup \{\mathbb{I} \times \{1/n\} : n \in \mathbb{N}\},$$

where each element of \mathcal{C} has the usual topology. Set

$$X_A = (A \times \{0\}) \cup (\cup\{\mathbb{I} \times \{1/n\} : n \in \mathbb{N}\}),$$

and let its topology be determined by the cover \mathcal{C} . It is known in [12, Theorem 2.2] that if a space is a quotient s -image of a locally separable, metric space, then it has a point-countable cs^* -network. Since X_A is a quotient s -image of the locally separable, metric space $\oplus\{C : C \in \mathcal{C}\}$, it has a point-countable cs^* -network. Since the subspaces $A \times \{0\}$ and $\bigcup_{n \in \mathbb{N}} \mathbb{I} \times \{1/n\}$ of X_A have the usual Euclidean topology, X_A has a countable network; in particular, X_A is hereditarily separable. It was proved in [6, Example 4.1(7)] that X_A has not any star-countable k -network. Therefore, X_A is not an \aleph_0 -space.

In general, X_A is not regular [17].

Definition 2.5 ([17]). A subset A of the real line is called a σ' -set if for every F_σ -set F of the real line, there is an F_σ -set H in the real line such that $F \cap H = \emptyset$ and $A \subset F \cup H$.

It is known that the Continuum Hypothesis implies the existence of an uncountable σ' -set [17].

Lemma 2.6. [17, Theorem 2.2] *Let A be a non-empty subset of \mathbb{I} . Then A is a σ' -set if and only if X_A is regular.*

Let A be an uncountable σ' -set in \mathbb{I} under the Continuum Hypothesis. By Lemma 2.6 and the observations on X_A , X_A is a counterexample to Question 1.5.

Finally, we remark that Question 1.6 is also negative. By using a σ' -set, Huaipeng Chen in [1, Example 4.5] constructed a space X satisfying the following properties.

- (1) X is regular T_1 , and a quotient s -image of a metric space;
- (2) every first countable closed subspace of X is locally compact;
- (3) X is not a compact-covering s -image of any metric space.

It was proved in [7, Theorem 1] that each quotient s -image of a locally compact, metric space is a compact-covering quotient s -image of a locally compact, metric space. By property (3) above, Chen's space X is not a quotient s -image of a locally compact, metric space since X is not a compact-covering quotient s -image of a locally compact, metric space. This contradicts property (3) above. Hence, X is a counterexample to Question 1.6.

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